Log-amplitude statistics of intermittent and non-Gaussian time series

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Abstract

In studies of hydrodynamic turbulence and nonequilibrium systems, it has been demonstrated that the observed non-Gaussian probability density functions are often described effectively by a superposition of Gaussian distributions with fluctuating variances. Based on this framework, we propose a general method to characterize intermittent and non-Gaussian time series. In our approach, an observed time series is assumed to be described by the multiplication of Gaussian and amplitude random variables, where the amplitude variable describes the variance fluctuation. It is shown analytically that statistical properties of the log-amplitude fluctuations can be estimated using the logarithmic absolute moments of the observed time series. This method is applicable to a wide variety of symmetric unimodal distributions with heavy tails in order to quantify the deviation from a Gaussian distribution. By analyzing random cascade-type processes and superstatistical non-Gaussian models with power-law tails, we demonstrate that our method can provide detailed characterization in a wide range of non-Gaussian fluctuations.

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I. INTRODUCTION

Recently, using statistical tools developed in hydrodynamic turbulence and nonequilibrium systems, numerous studies have been conducted to establish universal properties of intermittent fluctuations in a wide range of complex systems [1–8]. A remarkable property of intermittent fluctuations is inhomogeneity of variance, which results in non-Gaussian probability density functions (PDFs). To characterize such non-Gaussian fluctuations, it has been demonstrated that non-Gaussian PDFs are often described effectively by a superposition of Gaussian distributions with fluctuating variances [1, 2].

In the study of the velocity difference between two points in fully developed turbulent flows, Castaing *et al.* [1] introduced the following equation:

$$P(x) = \int_0^\infty \frac{1}{\sigma} P_L\left(\frac{x}{\sigma}\right) G(\ln\sigma)\, d(\ln\sigma), \tag{1}$$

where it is assumed that P_L is the standard Gaussian distribution and G is a distribution describing the fluctuation of the standard deviations. Under Kolmogorov's refined similarity hypothesis, G is assumed to be an infinitely divisible distribution. In their study, Castaing *et al.* focused mainly on the estimation of the variance of G and its scale behavior. This approach enables us to characterize a wide range of intermittent fluctuations, and has been applied in diverse fields such as econophysics [3], geophysics [4], and physiology [9]. However, in the previous studies, the estimation of the variance of G depends on a priori knowledge of the functional form of $G(\ln\sigma)$ such as log-normality, which would limit the applicability of this approach. The existing theories on intermittency in developed turbulence have predicted both shapes of G and scale dependence of its moments [1, 10]. Thus, it is required to develop a method for determining statistical properties of G from the observed data.

As pointed out by Jung and Swinney [11], equation (1) can be linked with Beck and Cohen's superstatistics [2] that has also been applied to a wide range of nonequilibrium systems (see, [2] and references therein). In addition, Friedrich *et al.* provided an exact solution of a generalized Kramers-Fokker-Planck equation [12], which can be given by superpositions of Gaussian distributions with varying variances. Hence, a general problem that is of great interest in experimental applications is how to objectively characterize the variance fluctuations described by G in Eq. (1). To solve this problem, we propose a novel method for estimating the variance and higher moments of G from the observed time series without

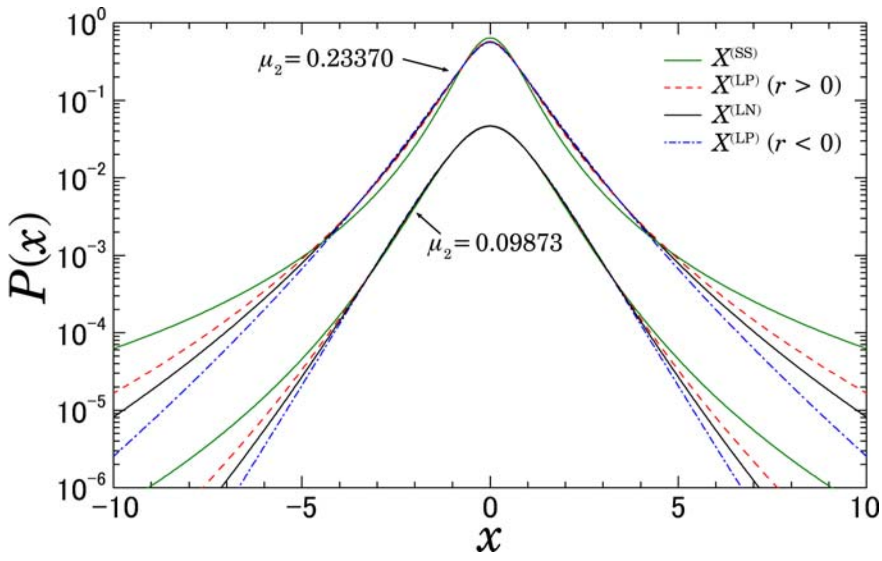

FIG. 1: Standardized PDFs of log-normal ($X^{(\mathrm{LN})}$), log-Poisson ($X^{(\mathrm{LP})}$), and superstatistical ($X^{(\mathrm{SS})}$) IID processes with the same log-amplitude variance μ_2, where $X^{(\mathrm{LN})}$, $X^{(\mathrm{LP})}$, and $X^{(\mathrm{SS})}$ are defined by Eqs. (7), (8), and (11), respectively. Solid lines (green): $X^{(\mathrm{SS})}$ with $k = 3$ (top) and $k = 6$ (bottom); dashed lines: $X^{(\mathrm{LP})}$ with $\lambda = 20$ and $r = \sqrt{\mu_2/\lambda}$; solid lines (black): $X^{(\mathrm{LN})}$ with $\alpha = \sqrt{\mu_2}$; dot-dashed lines: $X^{(\mathrm{LP})}$ with $\lambda = 20$ and $r = -\sqrt{\mu_2/\lambda}$. The PDFs are shifted in vertical directions for convenience of presentation.

assumptions on the shape of G [13]. In other words, we provide a systematic method to determine the shape of G based on only the observed time series.

Moreover, our method is applicable to a wide range of symmetric unimodal distributions with heavy tails, including symmetric power-law distributions, $P(x) \sim |x|^{-\alpha}$, with $1 < \alpha$. In general, the variance of G in Eq. (1) for a heavy tailed distribution can be interpreted as a measure of the deviation from a Gaussian distribution.

II. LOG-AMPLITUDE VARIANCE AND HIGHER MOMENTS

To explain our approach, let us assume that an observed stationary time series $\{x_i\}$ with zero mean is described by a multiplicative stochastic process,

$$X_i = W_i e^{Y_i}, \tag{2}$$

where W is a Gaussian random variable with zero mean, and Y is the other random variable independent of W. In this case, the PDF of X has the same functional form as Eq. (1), where $G(y)$ is the PDF of Y. Here, we refer to $\{Y_i\}$ as the log-amplitude fluctuation.

To characterize the log-amplitude fluctuation, we consider the variance and higher central moments of Y, $\mu_n = \langle (Y - \langle Y \rangle)^n \rangle$, where $\langle \cdot \rangle$ denotes the statistical average. By the calculation of the logarithmic absolute moments of X and the assumption of Eq. (2), we can obtain the following relations for μ_n:

$$\mu_2 = \left\langle (\ln|X| - \langle \ln|X| \rangle)^2 \right\rangle - \frac{\pi^2}{8}, \tag{3}$$

$$\mu_3 = \left\langle (\ln|X| - \langle \ln|X| \rangle)^3 \right\rangle + \frac{7}{4}\zeta(3), \tag{4}$$

$$\mu_4 = \left\langle (\ln|X| - \langle \ln|X| \rangle)^4 \right\rangle - \frac{3}{4}\pi^2\mu_2 - \frac{7}{64}\pi^4, \tag{5}$$

where $\zeta(n)$ is the Riemann zeta function ($\zeta(3) = 1.2020569\cdots$). Note that the logarithmic absolute moments of X do not depend on the variance of X, because

$$\ln|\sigma_0 X| - \langle \ln|\sigma_0 X| \rangle = \ln|X| - \langle \ln|X| \rangle, \tag{6}$$

where σ_0 is an arbitrary constant. Furthermore, we can obtain the estimator of the higher-order moment, if required.

Our key idea is to use logarithmic absolute moments in order to characterize the non-Gaussian PDF of X. If logarithmic absolute moments of X are finite, it is possible to define the log-amplitude moments. Even in the case where the PDF of X has power-law tails, $P(x) \sim |x|^{-\alpha}$, with $1 < \alpha$, the logarithmic absolute moments of X are finite, although the variance of X is undefined or infinite. Therefore, a wide range of symmetric unimodal distributions with heavy tails can be characterized by our approach. In other words, the log-amplitude variance μ_2 [Eq. (3)] is simply interpreted as the differnce in the second logarithmic-absolute moment between the observed non-Gaussian and Gaussian PDFs, because the second logarithmic-absolute moment of a Gaussian distribution equals $\pi^2/8$. Thus, μ_2 can be used as a measure of the deviation from a Gaussian distribution.

III. NUMERICAL EXAMPLES

To test our approach, we introduce illustrative examples of non-Gaussian stochastic processes and carry out numerical experiments.

A. Independent and identically distributed non-Gaussian variables

As a first step, we consider a stochastic process described by independent and identically distributed (IID) variables. The first example is a multiplicative log-normal process [14] based on experimental observations in the study of the turbulent velocity field [1], solar wind [4], foreign exchange rate [3], stock index [8], and human heartbeat [7, 15]. Neglecting the detailed structure of the intermittent dynamics, we mimic the PDFs observed in the log-normal processe. For comparison, we also introduce a multiplicative log-Poisson process. In the log-normal and log-Poisson processes, fluctuations of the standard deviations are assumed to obey log-normal and log-Poisson distributions, respectively.

In the log-normal process $\{X_i^{(\mathrm{LN})}\}$, the random variable $X^{(\mathrm{LN})}$ with zero mean is described by

$$X^{(\mathrm{LN})} = C_\alpha W e^{\alpha Y}, \tag{7}$$

where both W and Y are independent standard Gaussian random variables with zero mean and unit variance, and C_α is a scale parameter. In this process, the non-Gaussian nature is determined by the parameter α, which is the same as a non-Gaussian parameter λ defined in Ref. [14]. If $X^{(\mathrm{LN})}$ is standardized, C_α should be chosen as $C_\alpha = \exp(-\alpha^2)$. In this case, its log-amplitude moments are $\mu_2 = \alpha^2$, $\mu_3 = 0$ and $\mu_4 = 3\alpha^4$.

In the log-Poisson process $\{X_i^{(\mathrm{LP})}\}$, the random variable $X^{(\mathrm{LP})}$ is described by

$$X^{(\mathrm{LP})} = C_\lambda W e^{rP}, \tag{8}$$

where W are independent standard Gaussian random variables, P are independent Poisson random variables with mean λ and variance λ, r is a real valued parameter, and C_r is a scale parameter. If $X^{(\mathrm{LP})}$ is standardized, C_r should be chosen as $C_r = \exp\left(-\lambda\left(\exp(2r) - 1\right)/2\right)$. In this case, its log-amplitude moments are $\mu_2 = r^2\lambda$, $\mu_3 = r^3\lambda$, and $\mu_4 = r^4\lambda(1+3\lambda)$.

The next example is a stochastic process $\{X_i^{(\mathrm{SS})}\}$ based on so-called superstatistics [2]. Superstatistics considers an inhomogeneous driven nonequilibrium system that consists of many subsystems with different values of some intensive parameter β (e.g., the inverse effective temperature). Each subsystem is assumed to reach local equilibrium very quickly. In this case, if the local equilibrium distribution is Gaussian, we obtain

$$P(x) = \int_0^\infty \sqrt{\beta}\, P_L\left(\sqrt{\beta} x\right) f(\beta)\, d\beta, \tag{9}$$

where $P_L(x)$ is the standard normal distribution and $f(\beta)$ is the distribution of β. If $f(\beta)$ is a log-normal distribution, the PDF of superstatistics has the same form as the above log-normal process [11].

Here, we choose the χ^2 distribution with degree k,

$$f(\beta) = \frac{1}{\Gamma(k/2)} \left(\frac{k}{2\beta_0}\right)^{k/2} \beta^{\frac{k}{2}-1} e^{-\frac{k\beta}{2\beta_0}}, \tag{10}$$

which is one of the universality classes in superstatistics [2]. In this case, equation (9) results in the Student's t-distribution, which exhibits power-law tails, $P(x) \sim |x|^{-(k+1)}$ for large $|x|$. In this superstatistical process, the random variable $X^{(\mathrm{SS})}$ is described as

$$X^{(\mathrm{SS})} = W\sqrt{\frac{k}{\beta_0 Q}} \tag{11}$$

where W are independent standard Gaussian random variables, Q are independent χ^2 random variables with k degrees of freedom. When $k > 2$, $X^{(\mathrm{SS})}$ can be standardized by $\beta_0 = k/(k-2)$. In Castaing's description [Eq. (1)], the corresponding $G(y)$ is obtained as

$$G(y) = \frac{2}{\Gamma(k/2)} \left(\frac{k}{2\beta_0}\right)^{k/2} \exp\left(-ky - \frac{k}{2\beta_0} e^{-2y}\right). \tag{12}$$

Thus, $X^{(\mathrm{SS})}$ are described as $X^{(\mathrm{SS})} = We^Y$, where Y obey Eq. (12). In this case, its log-amplitude moments are $\mu_2 = \psi^{(1)}(k/2)/4$, $\mu_3 = -\psi^{(2)}(k/2)/8$, and $\mu_4 = (3\,\psi^{(1)}(k/2)^2 + \psi^{(3)}(k/2))/16$, where $\psi^{(n)}(x)$ is nth derivative of the Euler's psi function $\psi(x)$.

It is important to note that the log-amplitude moments μ_n can be defined for $k \geq 1$. When $k = 1$, the PDF of $X^{(\mathrm{SS})}$ reduces to a Cauchy distribution,

$$P(x) = \frac{\gamma}{\pi\,(x^2 + \gamma^2)}, \tag{13}$$

where the scale parameter γ is chosen as $\gamma = 1/\sqrt{\beta_0}$. Because this distribution has power-law tails, $P(x) \sim |x|^{-2}$, for large $|x|$, its second and higer moments are infinite. On the other hand, all of the log-amplitude moments μ_n are finite, which demonstrates that the log-amplitude statistics is applicable to a variety of heavy tailed distributions.

As shown in Fig. 1, the center parts of the PDFs have similar shapes, if the values of μ_2 are equal. To test the estimators μ_n, we numerically generate data sets using the above models, and then estimate the value of μ_n. To estimate the logarithmic absolute moments, $\langle(\ln|X| - \langle\ln|X|\rangle)^n\rangle$, in Eqs. (3)-(5), here we use the following estimators,

$$M_n = \frac{1}{N}\sum_{i=1}^{N} (\ln|X_i| - M_1)^n \quad (n = 2, 3, 4), \tag{14}$$

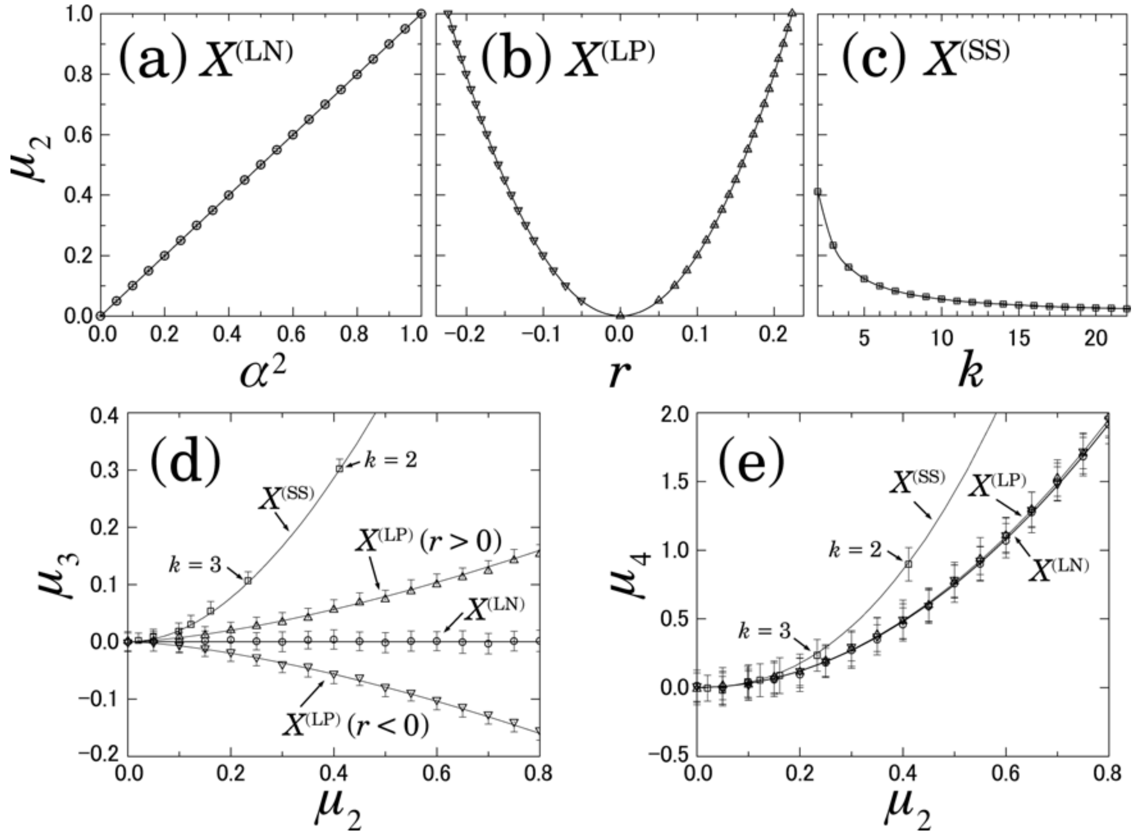

FIG. 2: Estimation of μ_n for log-normal ($X^{(\mathrm{LN})}$, circles), log-Poisson ($X^{(\mathrm{LP})}$, triangles), and superstatistical ($X^{(\mathrm{SS})}$, squares) IID processes, where $\lambda = 20$ for $X^{(\mathrm{LP})}$. The sample means of μ_n were estimated from 100 samples of length $N = 10^6$. The error bars indicate the sample standard deviation. The solid lines indicate the theoretical predictions.

where

$$M_1 = \frac{1}{N}\sum_{i=1}^{N} \ln |X_i|. \tag{15}$$

As shown in Fig. 2, the theoretical values are estimated well from the observed time series. In particular, in the plot of μ_3 vs μ_2 [Fig. 2 (d)], we find significant differences between the models, although in the plot of μ_4 vs μ_2 [Fig. 2 (e)] we find no differences between the log-normal and log-Poisson processes.

To evaluate the error of the estimate of μ_n, we study the mean squared error (MSE). The MSE of an estimator $\hat{\theta}$ of the parameter θ is defined as

$$\epsilon(\hat{\theta}) = \left\langle \left(\hat{\theta} - \theta\right)^2 \right\rangle. \tag{16}$$

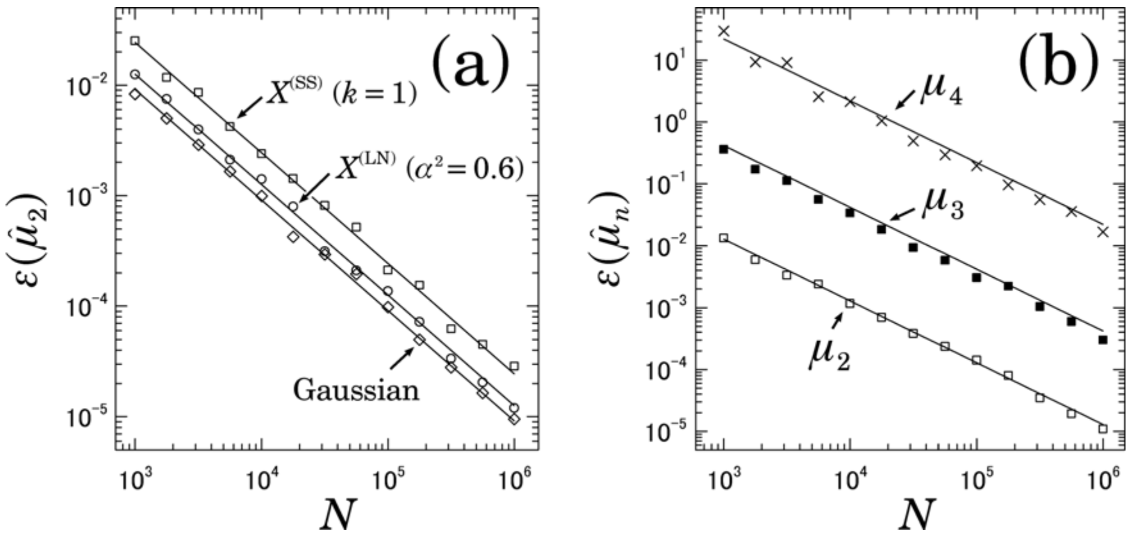

FIG. 3: Dependence of the mean squared error (MSE) on the data length N. (a) The MSE of μ_2 for log-normal ($X^{(\mathrm{LN})}$), superstatistical ($X^{(\mathrm{SS})}$), and Gaussian IID processes, where $\alpha^2 = 0.6$ for $X^{(\mathrm{LN})}$ and $k = 1$ for $X^{(\mathrm{SS})}$. The averages of the MSE for $X^{(\mathrm{LN})}$ (circles), $X^{(\mathrm{SS})}$ (squares), and Gaussian variables (diamonds) were computed from 200 samples. (b) The MSE of estimators of μ_n for $X^{(\mathrm{LN})}$ with $\alpha^2 = 0.6$. The averages of the MSE of μ_2 (open squares), μ_3 (filled squares), and μ_4 (crosses) were computed from 200 samples. The solid lines indicate the theoretical predictions [Eqs. (17) and (18)].

For the estimaotr of μ_2 and μ_3, the MSEs are evaluated as

$$\epsilon(\hat{\mu}_2) = \frac{1}{n}\left(\mu_4 - \mu_2^2 + \frac{\pi^2}{2}\mu_2 + \frac{3}{32}\pi^4\right) + O\left(\frac{1}{n^2}\right), \tag{17}$$

$$\begin{aligned}\epsilon(\hat{\mu}_3) &= \frac{1}{n}\left(\mu_6 + \frac{15}{8}\pi^2\mu_4 - \mu_3^2 + \frac{77}{2}\zeta(3)\mu_3 + \frac{105}{64}\pi^4\mu_2\right.\\ &\left.+\frac{441}{16}\zeta(3)^2 + \frac{139}{512}\pi^6\right) + O\left(\frac{1}{n^2}\right).\end{aligned} \tag{18}$$

The equation (17) implies that the statistical error of the estimate of μ_2 does not depend sensitively on the value of μ_2. As shown in Fig. 3 (a), the MSEs $\epsilon(\hat{\mu}_2)$ for Gaussian and Cauchy ($X^{(\mathrm{SS})}$ with $k = 1$) distributions are of the same order, although the value of μ_2 for the Cauchy distribution, $\mu_2 = 1.2337\cdots$, is considerably larger than $\mu_2 = 0$ for the Gaussian distribution. This is an advantage of our method compared to moment-based characterization using $\langle |X|^n \rangle$. As the order n of μ_n increases, the MSE $\epsilon(\hat{\mu}_n)$ rapidly increases. As shown in Fig. 3 (b), to estimate accurate values of the higher log-amplitude moments, very

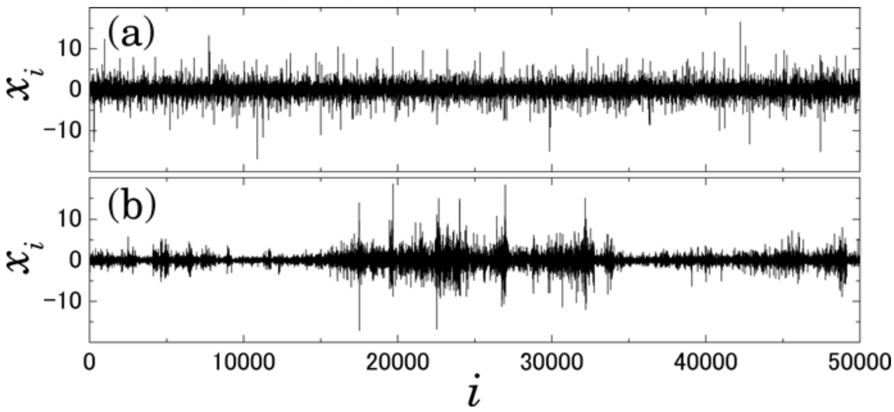

FIG. 4: Time series of log-normal processes. (a) a IID process [Eq. (7)] ($\alpha^2 = 0.6$); (b) a random cascade process [Eq. (19)] ($m = 16$, $\langle (Y^{(j)} - \langle Y^{(j)} \rangle)^2 \rangle = 0.6/m$).

large data sets are requied.

B. Random cascade-type processes

It is important to note that the characterization of the non-Gaussian PDF at a fixed scale is not sufficient to provide a deeper insight into intermittent dynamics. For instance, the time series shown in Fig. 4 (a) and (b) have the same PDF $P(x)$, although the properties of the variance inhomogeneity are quite different. A standard approach to characterize the intermittent time series is to describe how the shape of the PDF changes across scales. That is, from an observed time series $\{X_i\}$, consider the PDF of the partial sum $\Delta_s Z_i = \sum_{j=1}^{s} X_{i+j}$, where s indicates the scale. In other words, $\Delta_s Z_i$ is equal to the increment of the integrated series $Z_n = \sum_{i=1}^{n} X_i$. On the characterization of the PDF of $\Delta_s Z$, the existing theories have predicted the shapes of G in Eq. (1) and the scale dependence of the moments [1, 2, 10]. To discuss this point in more detail, we consider intermittent time series of multiplicative cascade processes.

Recently, we proposed a simple model of the cascade process, as described by Eq. (2) [14]. The numerical procedure to generate the time series is as follows: First, we generate a time series $\{W_i\}_{i=1}^{2m}$ of Gaussian white noise with zero mean and variance σ_0^2, where m is the total number of cascade steps. In the first cascade step ($j = 1$), we divide the whole interval into two equal subintervals, and then multiply W_i in each subinterval by random weights

$\exp[Y^{(1)}(k)]$ ($k = 0, 1$), where $Y^{(j)}$ are identical independent random variables. In the framework of Kolmogorov's refined similarity hypothesis, the PDF of $Y^{(j)}$ is assumed to be an infinitely divisible distribution $G_0(y)$. In the next cascade step ($j = 2$), we further divide each subinterval into two equal subintervals, and apply the random weights $\exp[Y^{(2)}(k)]$ ($k = 0, 1, 2, 3$). The same procedure is repeated, and after m cascade steps, the time series $\{X_i\}$ is given by

$$X_i = W_i \exp \sum_{j=1}^{m} Y^{(j)} \left(\left\lfloor \frac{i-1}{2^{m-j}} \right\rfloor \right), \tag{19}$$

where $\lfloor \cdot \rfloor$ is the floor function. If the PDF of $Y^{(j)}$ is an infinitely divisible distribution $G_0(y)$, the time series $\{X_i\}$ is described by the same form of Eq. (2) as $X_i = W_i \exp \overline{Y}_i^{(m)}$, where $\overline{Y}_i^{(m)} = \sum_{j=1}^{m} Y^{(j)}$ and its PDF is given by m-fold convolutions of $G_0(y)$. Moreover, if we approximate the distribution of the local sum of $\{X_i\}$ by a Gaussian, $\Delta_{s_n} Z$ at scale $s_n = 2^{m-n}$ is approximately given by $\Delta_{s_n} Z = \overline{W}^{(s_n)} \exp \overline{Y}^{(s_n)}$, where $\overline{W}^{(s_n)} = \sum_{k=1}^{s_n} W_k$ and $\overline{Y}^{(s_n)} = \sum_{j=1}^{n} Y^{(j)}$.

In the study of developed turbulence, one of the main statistical tools has been the multiscaling analysis of structure functions [10]. In our notations, the scaling of the structure functions $S_q(s)$ is described as $S_q(s) = \langle |\Delta_s Z|^q \rangle \sim s^{\zeta_q}$. In our model, the moments of $\Delta_{s_n} Z$ at scale $s_n = 2^{m-n}$ can be estimated as

$$\langle |\Delta_{s_n} Z|^q \rangle \approx \frac{(2\sigma_0^2)^{q/2}}{\sqrt{\pi}} \Gamma\left(\frac{q+1}{2}\right) e^{m\Psi(q)} s_n^{q/2 - \Psi(q)/\ln 2} \tag{20}$$

where $\Psi(q)$ is the cumulant-generating function of $Y^{(j)}$. Thus, this model has scaling exponents $\zeta_q = q/2 - \Psi(q)/\ln 2$. In this case, the log amplitude variance $\mu_2(s)$ of $\Delta_s Z_i$ is estimated as

$$\mu_2(s) \approx \mu_2^{(0)} (N - \log_2 s) \sim -\ln s, \tag{21}$$

where $\mu_2^{(0)}$ is the variance of $Y^{(j)}$. In general, the necessary condition of the existence of scaling exponents ζ_q for $\Psi(q) \neq 0$ is that variance and non-zero cumulants of $\overline{Y}^{(s)}$ are proportional to $-\ln s$.

Different from the logarithmic dependence, a power-law dependence of $\mu_2(s)$ has also been reported in experimental study [1, 4, 8]. As a phenomenological model exhibiting the power-law dependence, we assume a non-scale-invariant cascade process [16, 17], where the variance of $Y^{(j)}$ depends on the cascade step j as $\left\langle [Y^{(j)}]^2 \right\rangle \sim 2^{\gamma(j-1)}$. In this case, the scale dependence of $\mu_2(s)$ is estimated as

$$\mu_2(s) \approx \left[\frac{2^{\gamma m} s^{-\gamma} - 1}{2^{\gamma} - 1}\right] \mu_2^{(0)} \sim s^{-\gamma}. \tag{22}$$

To confirm the theoretical predictions, we numerically study log-normal and log-Poisson cascade processes, where $Y^{(j)}$ are Gaussian random variables and Poisson random variables multiplied by a real-valued parameter. A comparison between the log-normal and log-Poisson processes including an IID time series is shown in Fig. 5. In both cases, the scale dependences of μ_2 agrees well with the theoretical prediction [Fig. 5 (a-d)]. In addition, the deviation from log-normality is measured by μ_3. For the log-normal processes shown in Fig. 5 (e), the values of μ_3 are close to zero across the scales. Note that μ_3 of $\Delta_s Z$ $(s > 1)$ for the log-normal processes is not exactly equal to zero, because the PDFs of the log-normal processes is not stable distribution [14]. On the other hand, for log-Poisson processes shown in Fig. 5 (f), we can see the deviation from $\mu_3 = 0$ and the logarithmic dependence of μ_3 for the scale-invariant cascade process [Fig. 5 (f)].

IV. CONCLUSION

We proposed log-amplitude statistics to characterize non-Gaussian time series. Both turbulence statistics by Castaing *el al.* [1] and superstatistics by Beck and Cohen [2] have been very successful in describing non-Gaussian fluctuations. Including such examples, our method can be used to characterize non-Gaussian fluctuations. A crucial advantage in our approach is that a priori knowledge of the variance fluctuations is not assumed. The log-amplitude moments μ_n can provide a systematic way to quantify the shape of G in Eq. (1).

In addition, even in the case where the non-Gaussian PDF has power-law tails, $P(x) \sim |x|^{-\alpha}$, with $1 \leq \alpha \leq 2$, the the log-amplitude moments can be defined. Hence, our method is applicable to a wide range of symmetric unimodal distributions with heavy tails.

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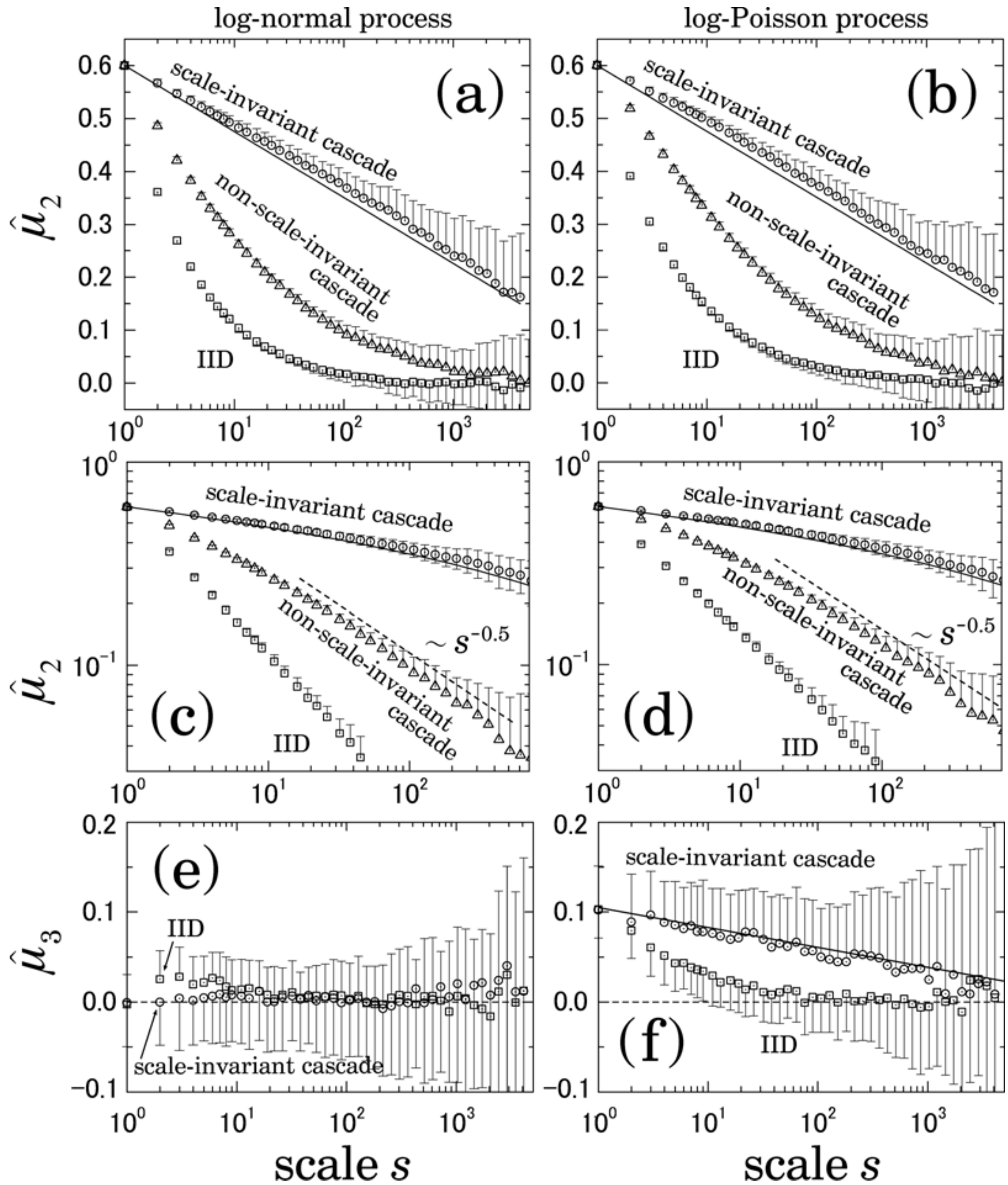

FIG. 5: Scale dependence of μ_2 and μ_3 for scale-invariant cascade (circles), non-scale-invariant cascade (triangles) and IID (squares) processes, where $\mu_2(1) = 0.6$ for all processes, $m = 16$ for cascade processes, and $\gamma = 0.5$ for non-scale-invariant cascade processes. The left and right panels show the log-normal and log-Poisson processes, respectively. The variance of Poisson random variables at $s = 1$ is $\lambda = 20$. The sample means of μ_n were estimated from 100 samples. The error bars indicate the sample standard deviation. The solid and dashed lines indicate the theoretical predictions.

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